JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 116, 222–229 (1986)

Characterizations of Almost Periodic Strongly Continuous Cosine Operator Functions

IOANA CIORANESCU

Fachbereich 17 Mathematik der Universität Paderborn, Warburgerstr. 100, 4790 Paderborn, West Germany

Submitted by Ky Fan

We characterize in terms of the infinitesimal generator those cosine operator functions which are almost periodic or uniformly almost periodic. © 1986 Academic Press, Inc.

Throughout this work $\{C_t\}_{t\in\mathbf{R}}$ is a strongly continuous operator function on the Banach space $\mathbf{X}$ with infinitesimal generator A. $\mathcal{N}(A)$ denotes the kernel of A and $\mathcal{R}(A)$ its range; $D(A)$ is the domain of A, $R(\cdot\,; A)$ its resolvent function and $\sigma(A)$ the spectrum.

DEFINITION. A strongly continuous cosine operator function $\{C_t\}_{t\in\mathbf{R}}$ is almost periodic, in short a.p., if for each $x\in\mathbf{X}$ the function $t\to C_t x$ from $\mathbf{R}$ to $\mathbf{X}$ is a.p.

$\{C_t\}_{t\in\mathbf{R}}$ is weakly almost periodic, in short w.a.p., if for each $x\in\mathbf{X}$ and $x^*\in\mathbf{X}^*$, the function $t\to\langle x^*, C_t x\rangle$ from $\mathbf{R}$ to $\mathbf{C}$ is a.

$\{C_t\}_{t\in\mathbf{R}}$ is uniformly almost periodic, in short u.: p., if the function $t\to C_t$ from $\mathbf{R}$ to $\mathcal{L}(\mathbf{X})$ is a.p.

Note that $\{C_t\}_{t\in\mathbf{R}}$ is u.a.p. if and only if the family of $\mathbf{X}$-valued functions $\{C_t x\}_{\|x\|\leqslant 1}$ is u.a.p.

Almost periodic and weakly almost periodic $\mathbf{X}$-valued functions were treated in [1, 3]; u.a.p. scalar functions are considered in [8] and $\mathbf{X}$-valued u.a.p. scalar functions are studied in [2, 10]. Characterizations of a.p., w.a.p. and u.a.p. groups and semi-groups of operators on Banach spaces are obtained by Bart and Goldberg in [2].

Our main result is:

THEOREM 1. *A strongly continuous cosine operator function $\{C_t\}_{t\in\mathbf{R}}$ is almost periodic if and only if $\{C_t\}_{t\in\mathbf{R}}$ is uniformly bounded and the set of eigenvectors of its infinitesimal generator A is total in $\mathbf{X}$.*

Proof. Let us first prove that the above conditions imply that $\{C_t\}_{t\in\mathbf{R}}$ is a.p.

0022-247X/86 $3.00

Recall first that the uniformly boundedness of $\{C_t\}_{t\in\mathbf{R}}$ implies that $\sigma(A)\subset\mathbf{R}$ (see [6, Satz 12] or [9, Theorem 3]).

For each $x\in\mathbf{X}$ and $\varepsilon>0$, there exist by assumption, $x_1, x_2,..., x_n$, eigenvectors of A, corresponding to eigenvalues $-\lambda_1^2, -\lambda_2^2,..., -\lambda_n^2$, $\lambda_j\in\mathbf{R}$, $1\leqslant j\leqslant n$, such that

$$\left\| x-\sum_{j=1}^{n}\alpha_j x_j\right\|\leqslant\varepsilon, \text{ for some } \alpha_j\in\mathbf{C},\ 1\leqslant j\leqslant n.$$

Note further that $C_t x_j=(\cos\lambda_j t)x_j$, $1\leqslant j\leqslant n$, $t\in\mathbf{R}$, and hence:

$$\left\| C_t x-\sum_{j=1}^{n}\alpha_j C_t x_j\right\|=\left\| C_t x-\sum_{j=1}^{n}\alpha_j(\cos\lambda_j t)x_j\right\|\leqslant M,$$

where $M=\sup_{t\in\mathbf{R}}\|C_t\|$.

Thus, for each $x\in\mathbf{X}$, $t\to C_t x$ is the uniform limit on $\mathbf{R}$ of continuous $\mathbf{X}$-valued a.p. functions and therefore is a.p.

Suppose now that $\{C_t\}_{t\in\mathbf{R}}$ is u.a.p.

From the fact that an a.p. function is bounded and with the uniform boundedness principle it follows that $\{C_t\}_{t\in\mathbf{R}}$ is uniformly bounded.

As $\{C_t\}_{t\in\mathbf{R}}$ is a.p., for each $x\in\mathbf{X}$ and $\lambda\in\mathbf{R}$, the "Fourier coefficient"

$$\begin{aligned} P_\lambda x &= \lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t} e^{-i\lambda s}C_s x\,ds \\ &= \lim_{t\to\infty}\frac{1}{t}\int_0^t \cos\lambda s C_s x\,ds \qquad (1)\end{aligned}$$

exists.

As $\|1/t\int_0^t\cos\lambda s\cdot C_s x\,ds\|\leqslant M\|x\|$, for each $x\in\mathbf{X}$, $t>0$, it is clear that $P_\lambda\in\mathscr{L}(\mathbf{X})$, $\lambda\in\mathbf{R}$.

For $\lambda, \tau\in\mathbf{R}$, $x\in\mathbf{X}$, we have

$$\begin{aligned} P_\lambda C_\tau x = C_\tau P_\lambda x &= \lim_{t\to\infty}\frac{1}{4t}\left[\int_{-t}^{t} e^{-i\lambda s}(C_{s+\tau}x+C_{s-\tau}x\right]ds \\ &= e^{i\lambda\tau}\lim_{t\to\infty}\frac{1}{4t}\int_{-t+\tau}^{t+\tau} e^{-i\lambda u}C_u x\,du \\ &\quad + e^{-i\lambda\tau}\lim_{t\to\infty}\frac{1}{4t}\int_{-t-\tau}^{t-\tau} e^{-i\lambda u}C_u x\,du \\ &= \frac{e^{i\lambda\tau}+e^{-i\lambda\tau}}{2}\cdot P_\lambda x=\cos\lambda\tau\cdot P_\lambda x,\end{aligned}$$

that is,

$$P_\lambda C_\tau x = C_\tau P_\lambda x = \cos \lambda\tau \cdot P_\lambda x, \qquad x \in \mathbf{X},\ \lambda,\ \tau \in \mathbf{R}. \tag{2}$$

But this implies

$$P_\lambda^2 x = \lim_{t \to \infty} \frac{1}{t} \int_0^t \cos \lambda\tau C_\tau P_\lambda x \, d\tau = \lim_{t \to \infty} \left(\frac{1}{t} \int_0^t \cos^2 \lambda\tau \, d\tau \right) P_\lambda x,$$

and hence

$$P_\lambda^2 = \begin{cases} P_\lambda & \text{for } \lambda = 0 \\ \frac{1}{2} P_\lambda & \text{for } \lambda \neq 0. \end{cases} \tag{3}$$

By [11, (2.12)] (see also [12, Proposition 2.3]), the operator $\tilde{A}$ defined by $\tilde{A}x = \lim_{t \to 0} (2/t^2)(C_t x - x)$, with domain those $x \in \mathbf{X}$ for which this limit exists, is the infinitesimal generator of $\{C_t\}_{t \in \mathbf{R}}$.

Then, using (2), we get, for $x \in \mathbf{X}$, $\lambda \in \mathbf{R}$,

$$\lim_{t \to 0} \frac{2}{t^2} (C_t - I) P_\lambda x = \lim_{t \to 0} 2 \, \frac{(\cos \lambda t - 1)}{t^2} P_\lambda x = -\lambda^2 P_\lambda x.$$

This implies that $P_\lambda x \in D(A)$ and that

$$(A + \lambda^2) P_\lambda x = 0, \qquad x \in \mathbf{X}, \lambda \in \mathbf{R}. \tag{4}$$

Suppose now that the set $\mathbf{E}$ of eigenvectors of A is not total in $\mathbf{X}$; then there exist $x^* \in \mathbf{X}^*$, $x^* \neq 0$, such that $\langle x^*, x \rangle = 0$, for each $x \in \mathbf{E}$.

But for each $\lambda \in \mathbf{R}$, $x \in \mathbf{X}$, $P_\lambda x$ is either 0 or in $\mathbf{E}$; in both cases $\langle x^*, P_\lambda x \rangle = 0$ and this implies that the a.p. scalar function $t \to \langle x^*, C_t x \rangle$ vanishes, because all its "Fourier coefficients" are zero. In particular, for $t = 0$, we get $\langle x^*, C_0 x \rangle = \langle x^*, x \rangle = 0$.

Since x is arbitrary in $\mathbf{X}$, this implies $x^* = 0$, a contradiction. ∎

Let us recall the following spectral characterization of the generators of uniformly bounded cosine operator functions, which is to be found in [11] or in [5]:

The closed and densely defined operator A is the infinitesimal generator of strongly continuous operator cosine function with $\|C_t\| \leqslant M$, $\forall t \in \mathbf{R}$, iff for all $z \in \mathbf{C}$ with $\operatorname{Re} z > 0$, $z^2 \in \mathbf{C} \backslash \sigma(A)$ and $\|d^n/dz^n[zR(z^2, A)]\| \leqslant M \cdot n!/(\operatorname{Re} z)^{n+1}$.

Thus we actually obtained with the above theorem a characterization of the almost periodicity of a cosine operator function solely in terms of the spectral properties of A.

PROPOSITION 2. *Let $\{C_t\}_{t\in\mathbf{R}}$ be a.p. and suppose that for $\lambda \in \mathbf{R}$, $-\lambda^2$ is an isolated point of $\sigma(A)$; then it is a simple pole for the resolvent $R(\cdot\,; A)$ with the residue M_λ, where $M_0 = P_0$ and $M_\lambda = 2P_\lambda$, for $\lambda \neq 0$ (P_λ are given by (1)).*

Proof. By (3) it is clear that M_λ are projections, $\forall \lambda \in \mathbf{R}$.
Moreover, we have

$$\mathcal{N}(A+\lambda^2 I) = \mathcal{R}(P_\lambda), \qquad \lambda \in \mathbf{R}. \tag{5}$$

In fact, the inclusion $\mathcal{R}(P_\lambda) \subset \mathcal{N}(A+\lambda^2)$ is given by (4).
In order to prove the converse inclusion, let us recall that in [9, Lemma 4], the following relation was shown:

$$AS(s,; a)x = a^2 S(s; a)x + a(C_s - \operatorname{ch} as)x, \qquad x \in \mathbf{X},\ a \in \mathbf{C}$$

where $S(s; a) = \int_0^s \operatorname{sh} a(s-t)\, C_t x\, dt$. Taking $a = i\lambda$, $\lambda \in R\setminus\{0\}$, we obtain

$$(C_s - \cos \lambda s)x = \frac{1}{i\lambda}(A+\lambda^2)\, S(s; i\lambda). \tag{6}$$

For $\lambda = 0$, by [11, (2.14)] we have

$$(C_s - I)x = A\int_0^s (s-t)\, C_t x\, dt. \tag{7}$$

Let $x \in \mathcal{N}(A+\lambda^2)$; then, from (6) and (7) we obtain

$$C_s x = (\cos \lambda s)x, \text{ for each } \lambda \in \mathbf{R}.$$

But now it follows

$$P_\lambda x = \lim_{t\to\infty} \frac{1}{t} \cos \lambda s\, C_s x\, ds$$

$$= \lim_{t\to\infty} \frac{1}{t}\int_0^t (\cos^2 \lambda s)\, x\, ds = \begin{cases} x & \text{for } \lambda = 0 \\ \dfrac{1}{2}x & \text{for } \lambda \neq 0 \end{cases}$$

such that $x \in \mathcal{R}(P_\lambda)$ and thus (5) is proved.
Let us further remark that as consequence of (2), we also have

$$\mathcal{R}(A+\lambda^2 I) \subset \mathcal{N}(P_\lambda) \tag{8}$$

(indeed, $P_\lambda Ax = \lim_{t\to\infty} 2P_\lambda((C_t - I)/t^2)x = \lim_{t\to\infty} 2((\cos \lambda t - 1)/\lambda^2)\, P_\lambda x = -\lambda^2 P_\lambda x, \forall x \in D(A)$).

Because of the relations (5) and (8) we can now follow the argument used in [2, Theorem 10] to prove a similar result for a.p. groups; we give the proof for completeness.

Let $-\lambda^2$ an isolated point in $\sigma(A)$ and P the spectral projection associated with A and $-\lambda^2$, then $\mathbf{X} = \mathcal{N}(P) \oplus \mathcal{R}(P)$ and A and C_t, $t \in \mathbf{R}$, are completely reduced by the above decomposition, that is $A = A_0 \oplus A_1$ and $C_t = C_t^0 \oplus C_t^1$, $\forall t \in \mathbf{R}$. Then $-\lambda^2 \notin \sigma(A_0)$ and $\sigma(A_1) = \{-\lambda^2\}$. As $\{C_t^1\}_{t\in\mathbf{R}}$ is a.p. with generator A_1 in $\mathcal{R}(P)$, by Theorem 1, the set of eigenvectors of A_1 is total in $R(P)$, that is, $\mathcal{N}(A_1 + \lambda^2) = \mathcal{R}(P)$, i.e., $A_1 = -\lambda^2 I \,|\, \mathcal{R}(P)$.

Thus $R(\cdot\,; A_1)$ has a simple pole at $-\lambda^2$ and this implies that also $R(\cdot\,; A) = R(\cdot\,; A_0) \oplus R(\cdot\,; A_1)$ has a simple pole at $-\lambda^2$.

To prove that $P = M_\lambda$, observe that by (5) we have

$$\begin{aligned}\mathcal{R}(P_\lambda) = \mathcal{N}(A + \lambda^2 I) &= \mathcal{N}(A_0 + \lambda^2 I) \oplus \mathcal{N}(A_1 \oplus \lambda^2 I)\\ &= \{0\} \oplus R(P) = \mathcal{R}(P).\end{aligned}$$

Finally, observe that

$$\begin{aligned}\mathcal{R}(A + \lambda^2 I) &= \mathcal{R}(A_0 + \lambda^2 I) \oplus \mathcal{R}(A_1 + \lambda^2 I)\\ &= \mathcal{N}(P) \oplus \{0\} = \mathcal{N}(P)\end{aligned}$$

such that (8) gives $\mathcal{N}(P) \subset \mathcal{N}(P_\lambda)$.

We obtained $\mathcal{R}(M_\lambda) = \mathcal{R}(P)$ and $\mathcal{N}(P) \subset \mathcal{N}(M_\lambda)$ and this implies $M_\lambda = P$ ∎

PROPOSITION 3. *If* $\mathbf{X}$ *is weakly sequentially complete, then each w.a.p. cosine operator function is a.p.*

Proof. Since each w.a.p. $\mathbf{X}$-valued function is bounded, the uniform boundedness of the cosine function $\{C_t\}_{t\in\mathbf{R}}$ results again with the uniform boundedness principle.

To prove that $\{C_t\}_{t\in\mathbf{R}}$ is a.p. it suffices, by Theorem 1, to prove that the set E of eigenvectors of A is total in $\mathbf{X}$.

Since X is weakly sequentially complete, for each $\lambda \in R$, $x \in \mathbf{X}$ and $x^* \in \mathbf{X}^*$, there is $y_{x,\lambda} \in \mathbf{X}$ such that

$$\lim_{t\to\infty} \left\langle x^*, \quad \frac{1}{t}\int_0^t \cos \lambda s C_s x\, ds \right\rangle = \langle x^*, y_{x,\lambda} \rangle.$$

Then, as before in the proof of (2), we get

$$\begin{aligned}\langle x^*, C_\tau y_{x,\lambda}\rangle &= \lim_{t\to\infty} \left\langle x^*, \frac{1}{2t}\int_{-t}^{t} e^{-i\lambda s} C_s C_\tau x\, ds\right\rangle \\ &= \lim_{t\to\infty} \left\langle x^*, e^{i\lambda\tau}/4t \int_{-t+\tau}^{t+\tau} e^{-i\lambda u} C_u x\, du\right. \\ &\quad \left. + e^{-i\lambda\tau}/4t \int_{-t-\tau}^{t-\tau} e^{-i\lambda u} C_u x\, du\right\rangle \\ &= \langle x^*, \cos\lambda\tau\, y_{x,\lambda}\rangle\end{aligned}$$

such that

$$C_\tau y_{x,\lambda} = \cos\lambda\tau\, y_{x,\lambda}.$$

But this implies again that $y_{x,\lambda} \in D(A)$ and $(A+\lambda^2)\, y_{x,\lambda} = 0$. We proceed further as in the proof of Theorem 1 with $y_{x,\lambda}$ at the place of $P_\lambda x$ and we obtain that E is total in $\mathbf{X}$. ∎

Remark. If $\mathbf{X}$ is a Hilbert space, then w.a.p. for a cosine operator function is equivalent to the a.p. and Theorem 5.1 from [4] gives us the structure of an a.p. cosine function, namely $C_t = \sum_{\lambda \geqslant 0} \cos\lambda t \cdot M_\lambda$, $t \in \mathbf{R}$.

We want, finally, to characterize the u.a.p. cosine operator functions and to this purpose we recall the following:

DEFINITION. A subset $\Lambda \subset \mathbf{R}$ is called harmonious if for each $\varepsilon > 0$ the set $D_\varepsilon = \bigcap_{\lambda\in\Lambda} \{t; |e^{i\lambda t} - 1| \leqslant \varepsilon\}$ is relatively dense in $\mathbf{R}$, in the sense that there exists and $l_\varepsilon > 0$ such that every interval of length l_ε meets D_ε.

For properties of harmonious sets and their role in the uniform almost periodicity of families of functions, see [8]. We have:

PROPOSITION 4. *A cosine operator function $\{C_t\}_{t\in\mathbf{R}}$ is u.a.p. iff it is a.p. and the set $\Lambda = \{\lambda \in \mathbf{R};\ -\lambda^2 \in \sigma(A)\}$ is harmonious.*

In this case $\sigma(A)$ consists only of simple poles of $R(\lambda; A)$.

Proof. Suppose that $\{C_t\}_{t\in\mathbf{R}}$ is u.a.p.; then, in particular, it is a.p. Let $\varepsilon > 0$; there exists, by assumption, a relatively dense subset $J_\varepsilon \subset \mathbf{R}$ such that for $t \in J_\varepsilon$, $\|C_t - I\| \leqslant \varepsilon$. Hence $\sigma(C_t) \subset \{\lambda \in \mathbf{C};\ |\lambda - 1| \leqslant \varepsilon\}$. By Theorem 3 in [9], we have: ch $t\sqrt{\sigma(A)} \subset \sigma(C_t)$, $t \in \mathbf{R}$ such that follows:

$$|\cos\lambda t - 1| = |\mathrm{ch}\, it\lambda - 1| \leqslant \varepsilon, \qquad t \in J_\varepsilon,\ \lambda \in \Lambda.$$

But this implies taht Λ is harmonious. Indeed

$$|e^{i\lambda t}-1|^2=|e^{i\lambda t}+e^{-i\lambda t}-2|\leqslant 2\varepsilon^2, \qquad t\in J_\varepsilon,\ \lambda\in\Lambda,$$

that is,

$$|e^{i\lambda t}-1|\leqslant\varepsilon \qquad \text{for} \quad t\in J_{\varepsilon^2/2} \text{ and } \lambda\in\Lambda.$$

As $J_{\varepsilon^2/2}$ is a relatively dense subset of $\mathbf{R}$, the assertion results.

Suppose now that $\{C_t\}_{t\in\mathbf{R}}$ is a.p. and that Λ is harmonious. We proved in Theorem 1, that each $x\in\mathbf{X}$ and $\lambda\in\Lambda$ with $P_\lambda x=\lim_{t\to\infty}(1/2t)\int_{-t}^{t}e^{-i\lambda s}C_s\,dt\neq 0$ has the property to be an eigenvector to eigenvalue $-\lambda^2$. But this means that the Fourier spectrum of each function $t\to C_t x$, $x\in\mathbf{X}$ is in Λ. As a family of uniformly bounded a.p. functions with their Fourier spectrum included in an harmonious set is u.a.p. (see [8, 2]) it follows that $\{C_t\}_{t\in\mathbf{R}}$ is u.a.p.

Finally, recall that if Λ is harmonious, then there exists $\delta>0$ such that the distance between two distinct elements in Λ exceeds δ ([8]); thus $\sigma(A)$ contains only isolated points and the last assertion follows from Proposition 2. ▮

For completeness we mention the following result due to Lutz [7]:

A cosine operator function is periodic with period 2π iff the following conditions are satisfied:

(i) $\sigma(A)\subset -\mathbf{N}_0^2$ and consists only of simple poles of $R(\lambda; A)$.

(ii) The set of eigenvectors of A is total in $\mathbf{X}$.

References

1. M. Amerio and G. Prouse, "Almost Periodic Functions and Functional Equations," Van Nostrand-Reinhold, New York, 1971.
2. H. Bart and S. Goldberg, Characterizations of almost periodic strongly continuous groups and semi-groups, *Math. Ann.* **236** (1978), 105–116.
3. C. Corduneanu, "Almost Periodic Functions," Interscience, New-York, 1968.
4. O. H. Fattorini, Uniformly bounded cosine functions in Hilbert space, *Indiana Univ. Math. J.* **20** (1970), 411–425.
5. O. H. Fattorini, Ordinary differential equations in linear topological spaces, I., *J. Differential Equations* **6** (1970), 50–70.
6. D. Lutz, Über operatorwertige Lösungen der Funktionalgleichung des Cosinus, *Math. Z.* **171** (1980), 233–245.
7. D. Lutz, Periodische operatorwertige Cosinusfunktionen, *Resultate Math.* **4** (1981), 75–83.
8. Y. Meyer, "Algebraic Numbers and Harmonic Analysis," North-Holland, Amsterdam, 1972.

9. B. Nagy, On cosine operator functions in Banach spaces, *Acta Sci. Math. (Szeged)* **36** (1974), 281–290.
10. W. M. Ruess and W. H. Summers, Compactness in spaces of vector valued continuous functions, preprint.
11. M. Sova, Cosine operator functions, *Rozprawy Mat.* **49** (1966).
12. C. C. Travis and G. F. Webb, Cosine families and abstract nonlinear second order differential equations, *Acta Math. Acod. Sci. Hungar.* **32** (3-4) (1978); 75–96.